Kernelization of embedded graphs

joined work with V. Delecroix and F. Lazarus

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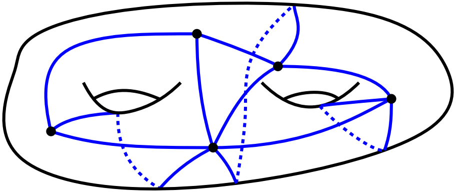
LABORATOIRE BORDELAIS DE RECHERCHE EN INFORMATIQUE LaBRI

INSTITUT FOURIER

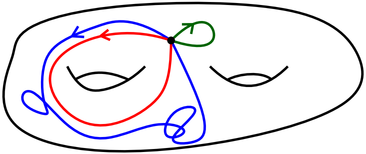
Combinatorial map

A **combinatorial map** is a graph $G = (V, E)$ cellularly embedded on a topological orientable compact surface S of genus g.
n is the number of vertices, e the number of edges and f the number of faces.
$n - e + f = 2 - 2g$.



Homotopy

Two closed loop γ and γ' on a surface are homotopic if one can be continuously deformed into the other.



- $\pi_1(S, v)$: group of loops based at v under homotopy
- $\mathcal{L}S$: set of loops under free homotopy

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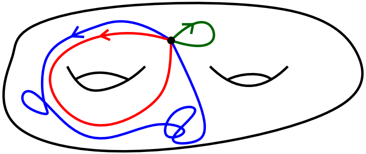

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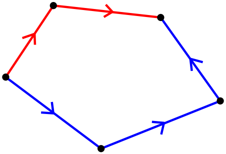
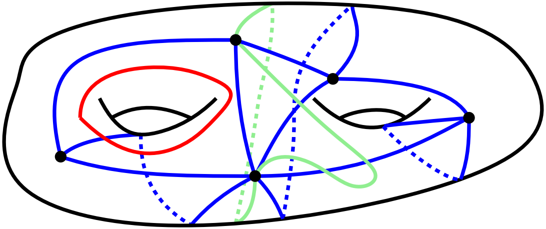
On a combinatorial map, homotopy is the closure of the following relation on faces :



Length spectrum

The **length spectrum** is the function $\mu_G : \begin{array}{ccc} \mathcal{LS} & \to & \mathbb{N} \\ \gamma & \mapsto & \inf_{\gamma' \sim \gamma} cr(\gamma, G) \end{array}$



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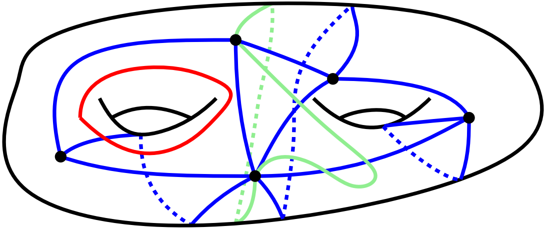


If H is a minor of G then $\mu_H \leqslant \mu_G$.
A **kernel** is an embedded graph G such that all proper minors H of G satisfy $\mu_H < \mu_G$.

Our result

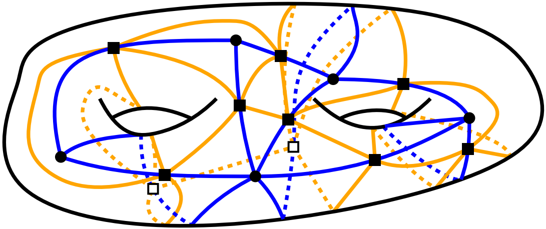
Theorem (Delecroix, F., Lazarus)

Given a map G of genus $g \geqslant 2$, a kernel minor can be computed in $O(n^3)$.

Medial graph

The medial graph M of G is defined by :

- one vertex on the middle of each edges
- one edge between the middles of two consecutive edges around each vertex



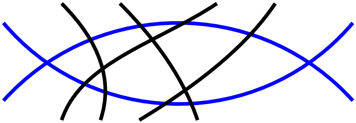
Universal covering

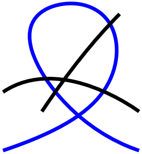
The **universal covering** $(\tilde{M}, p)$ of M on S is the tiling of the plane and a projection such that

- each face, edge or vertex of $\tilde{M}$ projects on a face, edge or vertex of M with same degree
- two adjacent faces of $\tilde{M}$ project on two adjacent faces of M
- two adjacent vertices of $\tilde{M}$ project on two adjacent vertices

Bigons and monogons

A **bigon** in M is the projection of a disk bounded by two lifts of curves in $\tilde{M}$.
A **monogon** in M is the projection of a disk bounded by a lift of curves in $\tilde{M}$.





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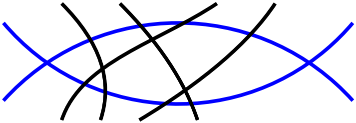

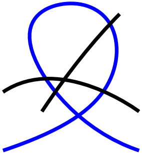

An **empty monogon** is a monogon without incident edges.
A **minimal bigon** is a bigon with no bigon nor monogon inside it.

Minimality

Theorem (Schrijver 1992)

G is not a kernel if and only if M contains either a minimal bigon or an empty monogon.

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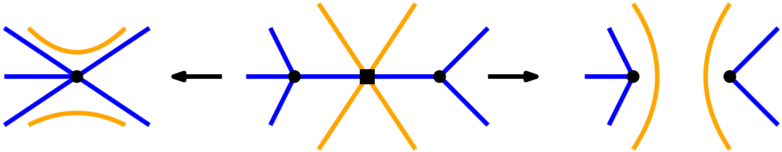
Moreover in that case a corner of such a bigon or monogon can be smoothed without changing the length spectrum.

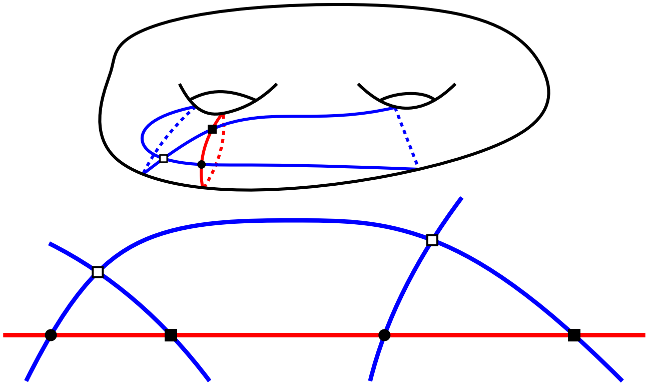


Length of a minimal bigon

Theorem (Delecroix, F., Lazarus)

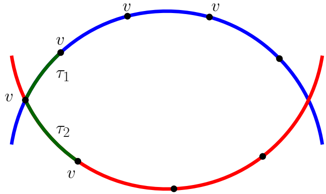
Assuming $g \geqslant 2$, the length of every minimal bigon in $\tilde{M}$ is at most $12n$.

Long bigon



Sketch of the proof

Goal : bound the number of lifts on one curve.



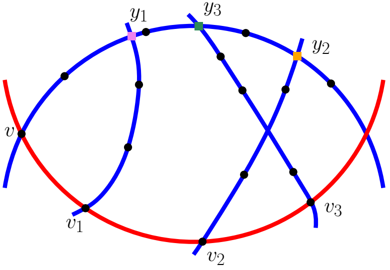
$$\pi_1(S, v) \simeq \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] = e \right\rangle$$

Look at the subgroup of $\pi_1(S, v)$ generated by τ_1 and τ_2. It is isomorphic to one of the following groups :

- $\{1\}$
- $\mathbb{Z}$
- F_2

A ping pong lemma

Assume $< \tau_1, \tau_2 > \simeq F_2$.



Area of a bigon

Theorem (Delecroix, F., Lazarus)

If $g \geqslant 2$, then the area of a bigon of length ℓ is at most $\dfrac{4g}{4g-6}\ell f$

Minimal bigon detection algorithm

Assume there is no empty monogon in M
for each vertex v of M **do**
    for all length $\ell \leqslant 12n$ **do**
        if there is a bigon b starting at v of length ℓ **then**
            Store the bigon
            if no bigon were found **then**
                G is a kernel
            else
                Among the stored bigon, pick one of minimal area

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This algorithm find a minimal bigon (if there is one) in $O(n^2)$

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- Testing if there is a bigon starting at v : homotopy test (Lazarus, Rivaud 2012 ; Colin de Verdière, Erickson 2010)
- Computing the area of a bigon

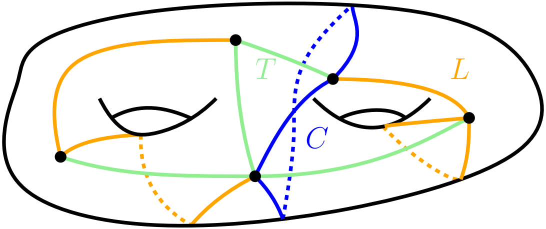
Tree/Co-tree decomposition

Theorem (Eppstein 2002)

There is a partition of the edges of G in three sets T, L, C such that :

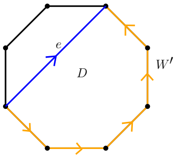
- *T is a spanning tree*
- *C is a co-spanning tree*
- *L has size $2g$*

Moreover such a partition can be computed in $O(n)$.



Idea of the algorithm

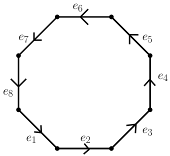
- Contracting the edge of T in M to have a 1-vertex map M'
- For each edge $e \in C$, compute the area between it and an homotopic path in L



- Compute the area of a contractible path on a map with 1 vertex and 1 face

Discrete Stokes formula

- $\overrightarrow{L} = \{e_1, e_2, \dots, e_{4g}\}$



- Choose two edges a and b in L
- Define "1-forms" ω and η as function $L \to \mathbb{N}$ such that :
  - $\omega(a) = \eta(b) = 1$ and $\omega(-a) = \eta(-b) = -1$
  - $\omega(e) = 0$ if $e \neq \pm a$ and $\eta(e) = 0$ if $e \neq \pm b$

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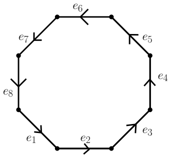

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  - $\omega(e) = 0$ if $e \neq \pm a$ and $\eta(e) = 0$ if $e \neq \pm b$
- $\omega \wedge \eta = \sum_{i<j} \omega(e_i)\eta(e_j) - \omega(e_j)\eta(e_i) = 2$
- $\sum_{D \in R} \omega \wedge \eta = \sum_{e \in \partial R} \alpha(e)\eta(e)$ for some potential α of ω

Conclusion

Algorithm to compute a kernel minor of G :

1. Compute the medial graph M of G
2. Find an empty monogon or a minimal bigon in M
3. Smooth one of its corners and do the corresponding minor operation on G
4. Repeat this operation until there is no monogon nor bigon

This process compute a kernel minor in $O(n^3)$.

Perspectives

- Length of any bigon ? Number of bigons ?
- Number of kernels with a given spectrum ?
- Compute the length spectrum